3.1 Exponential and Logistic Functions

**Exponential Function:**

\[ f(x) = a \cdot b^x \]

where: 
- \( a \) (initial value when \( x=0 \)) is a nonzero real number
- \( b \) (base) is positive, and \( b \neq 1 \)

Which are exponential functions?

a. \( f(x)=5^x \)  
   - **Yes**  
   - base = 5  
   - initial value = 1

b. \( f(x)=3x^2 \)  
   - **No**

c. \( f(x)=4x^{-3} \)  
   - **No**

d. \( f(x)=7 \cdot 2^{-x} \)  
   - **Yes**  
   - base = \( \frac{1}{2} \)  
   - initial value = 7
Determine the formulas for $g(x)$ and $h(x)$.

Values for Two Exponential Functions

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>$4/9$</td>
<td>128</td>
</tr>
<tr>
<td>-1</td>
<td>$4/3$</td>
<td>32</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

$$f(x) = a \cdot b^x$$

$$g(x) = 4 \cdot \left(\frac{3}{2}\right)^x$$

$$h(x) = 8 \cdot \left(\frac{1}{4}\right)^x$$
Exponential Growth and Decay

For any exponential function $f(x) = a \cdot b^x$ and any real number $x$,

- If $a > 0$ and $b > 1$, the function is increasing (exponential growth) and $b$ is the growth factor.
- If $a > 0$ and $b < 1$, the function is decreasing (exponential decay) and $b$ is the decay factor.
Exploration #1 (pg. 279)

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- Continuity: Continuous
- Increase/Decrease: $(-\infty, \infty)$
- Symmetry: None
- Boundedness: Unbounded/bounded below
- Extrema: None
- Asymptotes: $y = 0$
- End behavior:
Describe how each graph is transformed from \( f(x) = 5^x \).

a. \( g(x) = 5^{x+1} \) left 1

b. \( h(x) = 5^{-x} \) refl. over y

c. \( k(x) = 3 \cdot 5^x \) vert. stretch
\[
\left(1 + \frac{1}{x}\right)^x
\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>( (1+\frac{1}{x})^x )</th>
<th>( (1.1)^x )</th>
<th>( (1.01)^{100} )</th>
<th>( (1.001)^{1000} )</th>
<th>( (1.00001)^{100,000} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>(2.594)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>(2.705)</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>1000</td>
<td>(2.717)</td>
<td></td>
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<tr>
<td>10,000</td>
<td>(2.718)</td>
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<td>(2.718)</td>
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</table>
The Natural Base $e$. 

$$e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x$$
Exponential Functions and the Base e.

Any exponential function $f(x) = a \cdot b^x$ can be rewritten as

$f(x) = a \cdot e^{kx}$

for an appropriately chosen real number $k$.

If $a > 0$ and $k > 0$, $f(x)$ is an exp. growth fn.
If $a > 0$ and $k < 0$, $f(x)$ is an exp. decay fn.

Ex.
1. Graph $f(x) = 2^x$.

2. Overlay the graphs for $g(x) = e^{kx}$ for $k=0.4, 0.5, 0.6, 0.7, \text{ and } 0.8$.

3. For which value of $k$ does the graph of $g$ most closely match the graph of $f$?
Logistic Growth Functions:

\[ f(x) = \frac{c}{1 + a \cdot b^x} \quad \text{or} \quad f(x) = \frac{c}{1 + a \cdot e^{-kx}} \]

where \( a, b, c, \) and \( k \) are positive, with \( b < 1 \)

\( c \) is the limit to growth

If \( a = c = k = 1 \), then we get the logistic growth function

\[ f(x) = \frac{1}{1 + e^{-x}} \]
Graph and find the y-int. and horizontal asymptotes.

\[ f(x) = \frac{20}{1 + 2e^{-3x}} \]

**y-int. \( x = 0 \)**

\[ f(0) = \frac{20}{1 + 2e^{0}} = \frac{20}{3} = 6 \frac{2}{3} \]

**horiz. \( y = 20 \), \( y = 0 \)**
The number $B$ of bacteria in a petri dish culture after $t$ hours is given by

$$B = 100e^{0.693t}$$

a. What was the initial number of bacteria present?

b. How many bacteria are present after 6 hours?